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1996 J. Phys. A: Math. Gen. 29 709

(http://iopscience.iop.org/0305-4470/29/3/020)

J. Phys. A: Math. Gen. **29** (1996) 709–719. Printed in the UK

New solutions of the Yang–Baxter equation based on root of 1 representations of the para-Bose superalgebra $U_q[osp(1/2)]$

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Received 3 August 1995

Abstract. New solutions of the quantum Yang–Baxter equation, depending in general on three arbitrary parameters, are written down. They are based on the root of unity representations of the quantum orthosymplectic superalgebra $U_q[osp(1/2)]$, which were found recently. Representations of the braid group B_N are defined within any Nth tensorial power of root-of-1 $U_q[osp(1/2)]$ modules.

1. Introduction

In the present paper we write down new solutions of the quantum Yang–Baxter equation (QYBE), associated with root of unity representations of the quantum orthosymplectic superalgebra $U_q[osp(1/2)]$, which we have recently constructed [1]. All such representations are with a highest and a lowest weight. For q being a $4k$ root of 1 with $k = 3, 5, 7, \ldots,$ there exists a continuous class of k-dimensional representations. The solutions of the QYBE we find depend in general on three continuous parameters.

The general interest for studying solutions of the quantum Yang–Baxter equation is inspired from the various applications of the latter in conformal field theory [2, 3], quantum integrable models [4, 5] and knot theory [6–8]. Our motivation for the present investigation is of somewhat different nature. It originates from the close connection between the representations of the orthosymplectic superalgebras and the quantum statistics [9, 10], more precisely, the parastatistics [11].

It is perhaps worth commenting on the last point in greater detail. To this end consider as an example the Hopf algebra $U_q[osp(1/2n)]$, the quantized universal enveloping algebra of the orthosymplectic Lie superalgebra $osp(1/2n)$. The quantization of the latter in terms of its Chevalley generators is well known [12–17]. An alternative definition of $U_q[osp(1/2n)]$ has recently been given [18–21] in terms of pre-oscillator generators $a_i^\pm$, $K_i = q^{H_i}$, $i = 1, \ldots, n$. The relation to the quantum statistics stems from the observation that the operators $a_i^\pm$, $i = 1, \ldots, n$ can be identified with deformed para-Bose operators. Moreover, it turns out that the oscillator (or Weyl) superalgebra $W_q(n)$ generated by n pairs of deformed Bose operators [22–25] is a factor algebra of $U_q[osp(1/2n)]$ [26, 27] and (depending on the precise definition of the pre-oscillator generators) a morphism of $U_q[osp(1/2n)]$ onto $W_q(n)$ is given essentially by a replacement of the deformed para-Bose operators with deformed Bose operators. Therefore, despite the fact that the oscillator algebra $W_q(n)$

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0305-4470/96/030709+11$19.50

is not a Hopf algebra, one can define an R-matrix associated with $W_q(n)$ simply by considering the Fock representation of $W_q(n)$ also as a representation of $U_q[osp(1/2n)]$. To this end one has to express the $U_q[osp(1/2n)]$ universal R-matrix in terms of pre-oscillator generators and subsequently replace them with deformed Bose operators. The related matrices $R_{12},\ R_{13},\ R_{23}$, which are functions on n pairs of deformed Bose operators and the corresponding number operators, provide a 'bosonic' solution of the QYBE. Certainly one can try to carry out the above programme in a more general framework, considering other representations of the pre-oscillator generators. This would correspond to finding representations of the deformed para-Bose operators. The problem, however, is not simple; it has not been yet been solved even in the non-deformed case.

The present paper is a small step towards the realization of the above programme. Here we deal with the superalgebra $U_q[osp(1/2)]$. Nevertheless, even in this simple case one arrives at interesting conclusions. It turns out, for instance, that apart from the representations corresponding to both deformed and non-deformed parabosons (and, in particular, bosons) one finds a (root of 1) representation with $a^{\pm}$ being the usual fermions [28], i.e. the fermions are deformed parabosons. Thus, the bosons and the fermions appear as different irreps of one and the same quantized superalgebra, namely $U_q[osp(1/2)]$. As an example we write down the corresponding four-dimensional (non-diagonal) R-matrix, which leads to a 'fermionic' solution of the QYBE.

The new solutions of the QYBE will be based on the representations of the pre-oscillator generators $a^{\pm},\ K = q^{H}$ in (deformed para-Bose) Fock spaces. We pay special attention to the case when the deformation parameter q is a root of unity, which leads to finite-dimensional Fock spaces.

For $n > 1$ the pre-oscillator generators of $U_q[osp(1/2n)]$ are very different from its Chevalley generators. In case $n = 1$ however the creation and the annihilation (deformed para-Bose) operators $a^{+},\ a^{-}$ can be identified with the positive and the negative root vectors e and f of $U_q[osp(1/2)]$, respectively. Therefore the results to follow could have been given entirely in terms of the canonical terminology and notation for $U_q[osp(1/2)]$. We prefer, however, to stay close to the notation of the pre-oscillator generators, speaking about creation and annihilation operators instead of Chevalley generators, (deformed) Fock spaces instead of Verma modules, etc. In order to underline that $U_q[osp(1/2)]$ is (essentially) generated by deformed para-Bose operators we call it a (deformed) para-Bose superalgebra.

The paper is organized as follows. In section 2 we recall the definition of the deformed para-Bose superalgebra and its Fock irreps when q is a root of unity. The form of the transformation relations is new and more compact, compared with those given in [1]. In section 3 new solutions of the QYBE are constructed. The situation here is rather peculiar. We first prove that when q is a root of unity $U_q[osp(1/2)]$ is in general not almost cocommutative. Nevertheless, the expression of the (generic) universal R-matrix turns to be well defined within all of our representation spaces, which leads to solutions of the QYBE. In addition the R-matrix allows us to define representations of the braid group B_N in the Nth tensorial power of any of the $U_q[osp(1/2)]$ Fock modules.

Throughout we use the following abbreviations and notation: $\mathbb{C}$, all complex numbers; $\mathbb{Z}$, all integers; $\mathbb{Z}_+$, all non-negative integers; $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$; $[A, B] = AB - BA,\ \{A, B\} = AB + BA$; $U_q \equiv U_q[osp(1/2)]$.

2. The para-Bose algebra U_q[osp(1/2)] and its Fock irreps

Here we summarize the results of [1]. However, the form of the expressions (2.5), (2.7)–(2.9), describing the transformations of the Fock spaces, is new. It is more compact than the corresponding relations in [1].

The superalgebra $U_q = U_q[osp(1/2)],\ q \in \mathbb{C}\backslash\{0, \pm 1\}$ has three generators $a^+,\ a^-,\ H$, satisfying the defining relations:

$$[H, a^{\pm}] = \pm 2a^{\pm} \qquad \{a^+, a^-\} = \frac{q^H - q^{-H}}{q - q^{-1}}. \tag{2.1}$$

H is an even generator, $a^{\pm}$ are odd. As $q \to 1$ $H = \{a^+, a^-\}$ and equations (2.1) reduce to the defining relations of the non-deformed para-Bose operators [11] ($\xi, \eta, \epsilon = \pm$ or ± 1):

$$[\{a^{\xi}, a^{\eta}\}, a^{\epsilon}] = (\epsilon - \eta)a^{\xi} + (\epsilon - \xi)a^{\eta}. \tag{2.2}$$

The Hopf algebra structure on U_q can be defined in different ways [17]. For the comultiplication we set

$$\Delta(H) = H \otimes 1 + 1 \otimes H \qquad \Delta(a^+) = a^+ \otimes 1 + q^{-H} \otimes a^+ \qquad \Delta(a^-) = a^- \otimes q^H + 1 \otimes a^-. \tag{2.3}$$

Passing to the representations of U_q we note that the finite-dimensional irreps of $U_q[osp(1/2)]$ at generic q were constructed in [29, 30]. Some root of unity highest weight irreps were also obtained in [30]; both highest weight and cyclic representations were studied in [31–34].

A (deformed) Fock space $F(p)$ is defined in the usual way for the parastatistics [11]: for any complex p (which is an analogue of the order of the parastatistics) one postulates the existence of a vacuum vector $|0\rangle \in F(p)$ so that $a^-|0\rangle = 0$ and $H|0\rangle = p|0\rangle$. From now on we shall denote by $a_p^{\pm}$ and H_p the representatives of $a^{\pm}$ and H in $F(p)$. The latter is an infinite-dimensional linear space with a basis $|n\rangle = (a_p^+)^n|0\rangle,\ n \in \mathbb{Z}_+$.

Setting

$$\{n; x\}_q = \frac{q^{n+x} - (-1)^n q^{-n-x}}{q - (-1)^n q^{-1}} \tag{2.4}$$

one can write the transformation of the basis as follows:

$$H_p|n\rangle = (2n + p)|n\rangle \qquad a_p^-|n\rangle = \{n; 0\}_q\{n - 1; p\}_q|n - 1\rangle \qquad a_p^+|n\rangle = |n + 1\rangle. \tag{2.5}$$

At generic q the space $F(p)$ is infinite-dimensional. It is a simple (= irreducible) U_q module if p is not a negative even number [28] (which we always assume). The space $F(p = 1)$ is the Fock space of deformed Bose operators [22–25]. Within $F(1)$ the pre-oscillator operators satisfy the relations

$$a_1^- a_1^+ - q^{\pm 2} a_1^+ a_1^- = q^{\mp 2N} \qquad \text{where } N = \tfrac{1}{2}(H_1 - 1) \text{ is the number operator.} \tag{2.6}$$

In the root of unity cases $F(p)$ is indecomposible if and only if $q = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\frac{m}{k}}$ for every $m, k \in \mathbb{Z}$ such that $q \notin \{\pm 1, \pm \mathrm{i}\}$. The factor-space of $F(p)$ with respect to the maximal invariant subspace is an irreducible module, containing the vacuum vector $|0\rangle$.

The algebras U_q corresponding to all possible values of m and k contain several isomorphic copies. Without loss of generality we restrict m and k to values, which we call admissible, namely (i) $k = 2, 3, \ldots$; (ii) $m \in \{1, 2, \ldots, k - 1\}$; (iii) m and k are relatively co-prime. From now on we consider $q = e^{i\frac{\pi}{2}\frac{m}{k}}$ to be only an admissible root of 1.

The irreducible U_q modules with q being root of 1 are finite-dimensional. Denote by $W^L(p) \subset F(p)$ an $L+1$-dimensional representation space with a basis $|0\rangle,\ |1\rangle, \ldots, |L\rangle$. Its transformations under the action of the U_q generators read

$$\begin{aligned}
&H_p|n\rangle = (2n+p)|n\rangle \\
&a_p^-|n\rangle = \{n;0\}_q\{n-1;p\}_q|n-1\rangle \\
&a_p^+|L\rangle = 0 \\
&a_p^+|n\rangle = |n+1\rangle \qquad n < L.
\end{aligned} \tag{2.7}$$

We distinguish two classes of algebras, each one containing three groups of representations:

$$\begin{aligned}
\text{Class I } (k-m = \text{odd}):\ &\text{(I.a) } L = 2k-1 \text{ if } p \neq \text{integer;} \\
&\text{(I.b) } L = p(k-1)(\text{mod } 2k) \text{ if } p = \text{integer;} \\
&\text{(I.c) } L = 2k-1
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
\text{Class II } (k, m = \text{odd}):\ &\text{(II.a) } L = k-1 \text{ if } p \neq \text{even;} \\
&\text{(II.b) } L = (k-p)(\text{mod } k) \text{ if } p = \text{even;} \\
&\text{(II.c) } L = k-1.
\end{aligned} \tag{2.9}$$

The cases (I.a), (I.b), (II.a) and (II.b) correspond to irreducible representations, whereas in (I.c) ((II.c)) the representation is indecomposible if p = integer (p = even). The $2k$-dimensional modules corresponding to (I.c) were described in [32], where in particular it was shown how those of them corresponding to k = odd and m = even can be modified so that they carry cyclic representations. One has to keep in mind, however, that at certain values of p these modules are no longer irreducible, but are indecomposible. In fact each simple module $W^L(p)$ from (I.b) with $L = p(k-1)(\text{mod } 2k)$ is a factor-module of $W^{2k-1}(p)$ from (I.c) with respect to its maximal invariant subspace. To the best of our knowledge the representations from classes (I.b) and II have not so far been described in the literature.

One can always assume $0 < \text{Re}(p) \leqslant 4k$, since the representations with p outside that interval are equivalent to representations with p obeying the above inequality; if k is odd and m is even one can further set $0 < \text{Re}(p) \leqslant 2k$ if $m = 2(\text{mod } 4)$ and $0 < \text{Re}(p) \leqslant k$ if $m = 4(\text{mod } 4)$.

3. *R*-matrices and new solutions of the QYBE

One way for constructing R-matrices and hence solutions of the QYBE is based on the use of the universal R-matrix of a quasitriangular Hopf algebra U together with the representations of U.

The universal R-matrix for U_q was written down in [29, 30]. Here we use the expression as given in [17], which in our notation read

$$R = \sum_{n\geqslant 0} (-1)^{\frac{1}{2}n(n+1)} \frac{(q-\bar{q})^n}{(n)_{-\bar{q}^2}!} [(a^+)^n \otimes (a^-)^n] q^{\frac{1}{2}H\otimes H} \tag{3.1}$$

where $\bar{q} = q^{-1}$, $(n)_a = (1 - a^n)/(1 - a)$, $(n)_a! = (1)_a(2)_a \ldots (n)_a$.

If ρ_1 and ρ_2 are two representations of U_q defined in V_1 and V_2, then the related R-matrix is $R(\rho_1, \rho_2) = (\rho_1 \otimes \rho_2)R \in End(V_1 \otimes V_2)$.

In the root of 1 cases, however, the above construction generally fails, because for certain admissible q U_q is no longer almost cocommutative. The proof is essentially the same as the one given by Arnaudon for $U_q[sl(2)]$ [35]. It is based on the observation that U_q contains a larger centre generated from its Casimir operator and the additional central elements $\hat{x}^{\pm} = (a^{\pm})^{2k}$ and $\hat{z} = (K)^{2k}$ [30, 31]. If ρ is an irrep of U_q in V, then $\rho(\hat{x}^{\pm}) = \rho(x^{\pm})\mathbb{1}_V$, $\rho(\hat{z}) = \rho(z)\mathbb{1}_V$, where $\mathbb{1}_V$ is the unit operator in V and $\rho(x^{\pm}),\ \rho(z) \in \mathbb{C}$.

We proceed to show that the universal R-matrix does not exist for a subclass of I, corresponding to all algebras with $k =$ odd and $m =$ even. Let $N < 2k$ for $k - m =$ odd and $N < k$ for $k, m =$ odd. If $q = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\frac{m}{k}}$ and $AB + q^2BA = 0$ then the following general identity holds:

$$
\begin{aligned}
&(A+B)^N = \sum_{n=0}^{N} q^{-n(N-n)} \begin{Bmatrix} N \\ n \end{Bmatrix}_q A^n B^{N-n} \\
&\begin{Bmatrix} N \\ n \end{Bmatrix}_q = \frac{\{N\}_q!}{\{n\}_q!\{N-n\}_q!} \\
&\{n\}_q = q^n - (-1)^n q^{-n}.
\end{aligned}
\tag{3.2}
$$

Applying (3.2) for $N = 2k - 1$, $A = 1 \otimes a^-$ and $B = a^- \otimes K$, for all class I algebras we obtain:

$$
\Delta(\hat{x}^-) = 1 \otimes \hat{x}^- + \hat{x}^- \otimes \hat{z} \qquad \Delta^{\mathrm{op}}(\hat{x}^-) = \hat{x}^- \otimes 1 + \hat{z} \otimes \hat{x}^-. \tag{3.3}
$$

In (3.3) $\Delta^{\mathrm{op}} = \sigma\Delta$ is the opposite comultiplication; σ is a superpermutation, $\sigma(a \otimes b) = (-1)^{deg(a)deg(b)} b \otimes a$. Assume now that U_q is almost cocommutative, namely that there exists an invertible element R from (the completion of) $U_q \otimes U_q$, such that $R\Delta(a) = \Delta^{\mathrm{op}}(a)R$ for any $a \in U_q$. On the tensor product of two irreps ρ_1 and ρ_2 in V_1 and V_2 for $a = \hat{x}^-$ one would have:

$$
(\rho_1 \otimes \rho_2)(R)(\rho_1 \otimes \rho_2)(\Delta(\hat{x}^-)) = (\rho_1 \otimes \rho_2)(\Delta^{\mathrm{op}}(\hat{x}^-))(\rho_1 \otimes \rho_2)(R). \tag{3.4}
$$

With both sides of (3.4) acting on an arbitrary vector $|X\rangle \in V_1 \otimes V_2$, one gets

$$
\{\rho_1(x^-) + \rho_1(z)\rho_2(x^-)\}|Y\rangle = \{\rho_2(x^-) + \rho_2(z)\rho_1(x^-)\}|Y\rangle
$$

where $|Y\rangle = (\rho_1 \otimes \rho_2)(R)|X\rangle$. Therefore

$$
\rho_2(x^-) + \rho_2(z)\rho_1(x^-) = \rho_1(x^-) + \rho_1(z)\rho_2(x^-). \tag{3.5}
$$

The central elements $\hat{x}^{\pm}$ and $\hat{z}$ can take arbitrary values on the cyclic irreps of the algebras with $k =$ odd and $m =$ even [34], i.e. in this case $\rho_1(x^-)$, $\rho_2(x^-)$, $\rho_1(z)$ and $\rho_2(z)$ are arbitrary numbers, which contradicts (3.5). Therefore the universal R-matrix cannot exist for these algebras. Note, however, that equation (3.5) does not contradict the representations (2.8), since for any of them $\rho(x^{\pm}) = 0$. Therefore, following Rosso [36], one can try to produce an almost universal R-matrix on the quotient $\tilde{U}_q = U_q[osp(1/2)]/(\hat{x}^{\pm} = 0)$.

Equation (3.5) holds for the subclass of the class I algebras, corresponding to $k =$ even and $m =$ odd. The known irreps for this subclass are only those listed in (2.8). The latter do not contradict (3.5), since $\hat{x}^{\pm}$ act as zero operators within each class I U_q-module. Therefore the question as to whether the universal R matrix exists for the algebras with $k =$ even and $m =$ odd is an open one. The same holds for all algebras from the class II. Within each U_q module corresponding to (2.9) $\hat{x}^{\pm}$ are zero operators. Our attempts to extend these modules to carry cyclic representations were not successful. Moreover, equations (3.3), and hence equations (3.5) are no longer true.

We see that the question about the existence of an universal R matrix for the algebras U_q when q is a root of 1 cannot be answered uniquely at present. Our claim is that the R-matrix (3.1), considered as an element of $\tilde{U}_q \otimes \tilde{U}_q$, is almost universal, namely it is 'universal' for all Fock representations (2.7)–(2.9): if $\rho^{L_1}(p_1)$ and $\rho^{L_2}(p_2)$ are any two such representations, then the operator

$$R^{L_1,L_2}(p_1, p_2) = (\rho^{L_1}(p_1) \otimes \rho^{L_2}(p_2))(R) : \; W^{L_1}(p_1) \otimes W^{L_2}(p_2) \to W^{L_1}(p_1) \otimes W^{L_2}(p_2) \tag{3.6}$$

satisfies the analogue of (3.4)

$$R^{L_1,L_2}(p_1, p_2)(\rho^{L_1}(p_1) \otimes \rho^{L_2}(p_2))(\Delta(a)) = (\rho^{L_1}(p_1) \otimes \rho^{L_2}(p_2))(\Delta^{\mathrm{op}}(a))R^{L_1,L_2}(p_1, p_2). \tag{3.7}$$

The explicit action of $R^{L_1,L_2}(p_1, p_2)$ on the basis $|l_1\rangle \otimes |l_2\rangle$ of $W^{L_1}(p_1) \otimes W^{L_2}(p_2)$ yields

$$\begin{aligned} R^{L_1,L_2}(p_1, p_2)(|l_1\rangle \otimes |l_2\rangle) &= q^{\frac{1}{2}(2l_1+p_1)(2l_2+p_2)} \sum_{n=0}^{\min(L_1-l_1,l_2)} (-1)^{\frac{n}{2}(n+2l_1+1)} \frac{(q-\bar{q})^n}{(n)_{-\bar{q}^2}!} \\ &\quad \times \prod_{i=0}^{n-1} \{l_2 - i; 0\}_q \{l_2 - 1 - i; p_2\}_q |l_1 + n\rangle \otimes |l_2 - n\rangle. \end{aligned} \tag{3.8}$$

The proof of (3.7) is by a direct computation within each U_q module $W^{L_1}(p_1) \otimes W^{L_2}(p_2)$, i.e. using the transformation relations (3.8).

The linear operators

$$\begin{aligned} &R_{12}^{L_1,L_2}(p_1, p_2), R_{13}^{L_1,L_3}(p_1, p_3), R_{23}^{L_2,L_3}(p_2, p_3) \text{ in } W^{L_1,L_2,L_3}(p_1, p_2, p_3) \\ &\qquad \equiv W^{L_1}(p_1) \otimes W^{L_2}(p_2) \otimes W^{L_3}(p_3) \end{aligned} \tag{3.9}$$

which satisfy the QYBE

$$R_{12}^{L_1,L_2}(p_1, p_2)R_{13}^{L_1,L_3}(p_1, p_3)R_{23}^{L_2,L_3}(p_2, p_3) = R_{23}^{L_2,L_3}(p_2, p_3)R_{13}^{L_1,L_3}(p_1, p_3)R_{12}^{L_1,L_2}(p_1, p_2). \tag{3.10}$$

are defined on the basis as follows:

$$R_{12}^{L_1,L_2}(p_1, p_2)(|l_1\rangle \otimes |l_2\rangle \otimes |l_3\rangle) = q^{\frac{1}{2}(2l_1+p_1)(2l_2+p_2)} \sum_{n=0}^{\min(L_1-l_1,l_2)} (-1)^{\frac{n}{2}(n+2l_1+1)}$$

$$\times \frac{(q-\bar{q})^n}{(n)_{-\bar{q}^2}!}\prod_{i=0}^{n-1}\{l_2-i;0\}_q\{l_2-1-i;\, p_2\}_q|l_1+n\rangle\otimes|l_2-n\rangle\otimes|l_3\rangle. \tag{3.11}$$

$$R_{13}^{L_1,L_3}(p_1,\, p_3)(|l_1\rangle\otimes|l_2\rangle\otimes|l_3\rangle) = q^{\frac{1}{2}(2l_1+p_1)(2l_3+p_3)}\sum_{n=0}^{\min(L_1-l_1,l_3)}(-1)^{\frac{n}{2}(n+2l_1+2l_2+1)}$$
$$\times \frac{(q-\bar{q})^n}{(n)_{-\bar{q}^2}!}\prod_{i=0}^{n-1}\{l_3-i;0\}_q\{l_3-i-1;\, p_3\}|l_1+n\rangle\otimes|l_2\rangle\otimes|l_3-n\rangle. \tag{3.12}$$

$$R_{23}^{L_2,L_3}(p_2,\, p_3)(|l_1\rangle\otimes|l_2\rangle\otimes|l_3\rangle) = q^{\frac{1}{2}(2l_2+p_2)(2l_3+p_3)}\sum_{n=0}^{\min(L_2-l_2,l_3)}(-1)^{\frac{n}{2}(n+2l_2+1)}$$
$$\times \frac{(q-\bar{q})^n}{(n)_{-\bar{q}^2}!}\prod_{i=0}^{n-1}\{l_3-i;0\}_q\{l_3-i-1;\, p_3\}|l_1\rangle\otimes|l_2+n\rangle\otimes|l_3-n\rangle. \tag{3.13}$$

The operators (3.9) can be expressed in terms of the R-matrix (3.8). To this end introduce a superpermutation linear operator $P_{23} : (|n_1\rangle\otimes|n_2\rangle)\otimes|n_3\rangle) = (-1)^{n_2n_3}|n_1\rangle\otimes|n_3\rangle\otimes|n_2\rangle)$. Then

$$\begin{aligned} R_{12}^{L_1,L_2}(p_1,\, p_2) &= R^{L_1,L_2}(p_1,\, p_2)\otimes 1\\ R_{13}^{L_1,L_3}(p_1,\, p_3) &= P_{23}(R^{L_1,L_3}(p_1,\, p_3)\otimes 1)P_{23}\\ R_{23}^{L_2,L_3}(p_2,\, p_3) &= 1\otimes R^{L_2,L_3}(p_2,\, p_3). \end{aligned} \tag{3.14}$$

Depending on the choice of the representations (2.8) and (2.9), one obtains R-matrices of different dimensions, which may be parameter independent or can depend on one or two free parameters.

If $\rho^{L_1}(p_1),\ \rho^{L_2}(p_2) \in$ (I.c), then $R^{L_1,L_2}(p_1,\, p_2)$ depends on two arbitrary complex parameters p_1 and p_2, $\dim(R^{L_1,L_2}(p_1,\, p_2)) = 4k^2$. These R-matrices were obtained in [32]. The expression (3.8) is somewhat more compact.

If $\rho^{L_1}(p_1),\ \rho^{L_2}(p_2) \in$ (II.c) $R^{L_1,L_2}(p_1,\, p_2)$ depends also on the arbitrary complex parameters p_1 and p_2, but $\dim(R^{L_1,L_2}(p_1,\, p_2)) = k^2$. This is a new class of R-matrices, leading through (3.14) to new solutions of the QYBE, defined in a k^3-dimensional space $W^{L_1,L_2,L_3}(p_1,\, p_2,\, p_3)$ with $k = 3, 5, 7, \ldots$ and depending on three arbitrary parameters.

In all other cases the R-matrices depend on less then two free parameters, which is due to the case that for certain values of $p_1,\ p_2$ and p_3 $W^{L_1,L_2,L_3}(p_1,\, p_2,\, p_3)$ contains invariant subspaces. Those corresponding to $\rho^{L_1}(p_1),\ \rho^{L_2}(p_2) \in$ (I.b) or (II.b) lead to constant R-matrices and hence to constant solutions of the QYBE. Here are two examples.

Example 1. The representation (I.b) with $k = 2,\ (m = 1)$ and $p = 1$ gives $L = 1$. From (2.7) one concludes that $a^{\pm}$ are Fermi operators. In the basis $\{|0\rangle\otimes|0\rangle,\ |0\rangle\otimes|1\rangle,\ |1\rangle\otimes|0\rangle,\ |1\rangle\otimes|1\rangle\}$ the 'fermionic' R-matrix reads

$$R^{L_1=1,L_2=1}(p_1=1,\, p_2=1) = \begin{pmatrix} e^{\frac{1}{8}i\pi} & 0 & 0 & 0\\ 0 & e^{\frac{3}{8}i\pi} & 0 & 0\\ 0 & e^{\frac{1}{8}i\pi}-e^{\frac{5}{8}i\pi} & e^{\frac{3}{8}i\pi} & 0\\ 0 & 0 & 0 & -e^{\frac{1}{8}i\pi} \end{pmatrix}. \tag{3.15}$$

It contains no free parameters.

Example 2. We consider the class II algebra U_q with the smallest possible value of k, namely $k = 3$ (and hence $m = 1$), i.e. $q = e^{i\pi/6}$. There is a tree of R-matrices, related to the different possible branches of the representations (II.a, b, c). One such branch is, for instance, $R^{2,2}(p_1, p_2) \to R^{2,1}(p_1, 2) \to R^{1,1}(2, 2)$. The root R-matrix $R^{L_1=2,L_2=2}(p_1, p_2)$ is nine-dimensional and depends on two arbitrary parameters p_1 and p_2. In a matrix form (ordering the basis lexically, $|i\rangle \otimes |j\rangle < |k\rangle \otimes |l\rangle$ if $i < k$ or if $i = k$ and $j < l$) from (3.8) one obtains

$$R^{2,2}(p_1, p_2) = \begin{pmatrix} A_{00,00} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{01,01} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{02,02} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{10,01} & 0 & A_{10,10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{11,02} & 0 & A_{11,11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{12,12} & 0 & 0 & 0 \\ 0 & 0 & A_{20,02} & 0 & A_{20,11} & 0 & A_{20,20} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{21,12} & 0 & A_{21,21} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22,22} \end{pmatrix}. \quad (3.16)$$

with

$$A_{00,00} = e^{\frac{1}{12} i\pi p_1 p_2} \qquad A_{01,01} = e^{\frac{1}{12} i\pi p_1 (p_2+2)} \qquad A_{02,02} = e^{\frac{1}{12} i\pi p_1 (p_2+4)}$$

$$A_{10,10} = e^{\frac{1}{12} i\pi (p_1+2) p_2} \qquad A_{11,11} = e^{\frac{1}{12} i\pi (p_1+2)(p_2+2)} \qquad A_{12,12} = e^{\frac{1}{12} i\pi (p_1+2)(p_2+4)}$$

$$A_{20,20} = e^{\frac{1}{12} i\pi (p_1+4) p_2} \qquad A_{21,21} = e^{\frac{1}{12} i\pi (p_1+4)(p_2+2)} \qquad A_{22,22} = e^{\frac{1}{12} i\pi (p_1+4)(p_2+4)}$$

$$A_{10,01} = -2ie^{\frac{1}{12} i\pi p_1 (p_2+2)} \sin(\tfrac{1}{6}\pi p_2) \qquad A_{11,02} = -2ie^{\frac{1}{12} i\pi p_1 (p_2+4)} \cos(\tfrac{1}{6}\pi (p_2+1))$$

$$A_{20,11} = 2ie^{\frac{1}{12} i\pi (p_1+2)(p_2+2)} \sin(\tfrac{1}{6}\pi p_2)$$

$$A_{20,02} = -ie^{\frac{1}{12} i\pi (p_1(p_2+4)+2)} (2\sin(\tfrac{1}{6}\pi(2p_2+1)) - 1)$$

$$A_{21,12} = 2ie^{\frac{1}{12} i\pi (p_1+2)(p_2+4)} \cos(\tfrac{1}{6}\pi (p_2+1)).$$

Setting $p_2 = 2$ and $L_2 = 1$ one obtains the next matrix from the branch, namely the six-dimensional R-matrix $R^{L_1=2,L_2=1}(p_1, p_2 = 2)$, which depends on the arbitrary parameter p_1:

$$R^{2,1}(p_1, 2) = \begin{pmatrix} e^{\frac{1}{6} i\pi p_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\frac{1}{3} i\pi p_1} & 0 & 0 & 0 & 0 \\ 0 & -i\sqrt{3}e^{\frac{1}{3} i\pi p_1} & e^{\frac{1}{6} i\pi (p_1+2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\frac{1}{3} i\pi (p_1+2)} & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{3}e^{\frac{1}{3} i\pi (p_1+2)} & e^{\frac{1}{6} i\pi (p_1+4)} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\frac{1}{3} i\pi (p_1+4)} \end{pmatrix}. \quad (3.17)$$

$R^{2,1}(p_1, 2)$ can be obtained from the root matrix (3.16) by crossing out its rows and columns with numbers 3, 6 and 9 and setting $p_2 = 2$.

The last matrix from the branch corresponds to $p_1 = p_2 = 2$ and $L_1 = L_2 = 1$. It is a four-dimensional constant R-matrix, which can be obtained by crossing out the last two rows and columns in (3.17) and setting $p_1 = 2$:

$$R^{1,1}(2,2) = \begin{pmatrix} e^{\frac{1}{3}i\pi} & 0 & 0 & 0 \\ 0 & e^{\frac{2}{3}i\pi} & 0 & 0 \\ 0 & -i\sqrt{3}e^{\frac{2}{3}i\pi} & e^{\frac{2}{3}i\pi} & 0 \\ 0 & 0 & 0 & -e^{\frac{1}{3}i\pi} \end{pmatrix}. \tag{3.18}$$

One can choose certainly other branches from the R-matrix tree, obtaining in this way new R-matrices of smaller dimensions, which are always submatrices of the root matrix (3.16).

Let us mention at the end, following Zhang [38], that the R-matrix can be used also in order to define representations of the braid group B_N acting in any Nth tensorial power of Fock spaces $W^L(p)$, namely in $W^L(p)^{\otimes N}$. To this end set $\check{R}^L(p) = PR^{L,L}(p,p) \in End(W^L(p) \otimes W^L(p))$, where P is the superpermutation operator in $W^L(p) \otimes W^L(p)$. It is straightforward to verify that $\check{R}^L(p)$ is an $U_q[osp(1/2)]$ intertwining operator in $W^L(p) \otimes W^L(p)$:

$$[\check{R}^L(p), \Delta(a)] = 0 \qquad \forall\ a \in U_q. \tag{3.19}$$

Hence [38] $\sigma_i \in End(W^L(p))^{\otimes N}$ $\;i = 1, \ldots, N-1$, defined as

$$\sigma_i = \mathbb{1}^{\otimes(i-1)} \otimes \check{R}^L(p) \otimes \mathbb{1}^{\otimes(N-i-1)} \tag{3.20}$$

gives a representation of B_N, namely the $\sigma_1, \ldots, \sigma_{N-1}$ satisfy the defining relations for B_N:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j| > 1 \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}. \tag{3.21}$$

Hence (the representation of the braid group) B_N is a subset of the set of all intertwining operators in $W^L(p)^{\otimes N}$.

4. Concluding remarks

We have found new solutions of the quantum Yang–Baxter equations, using essentially the representations of $U_q[osp(1/2)]$, which we have recently constructed. The solutions were obtained formally from the 'generic' R-matrix (3.1), despite the fact that the latter does not exist in root-of-1 cases. The more precise statement is that at values of the deformation parameter $q = e^{i\frac{\pi}{2}\frac{m}{k}}$ with $k =$ odd and $m =$ even $U_q[osp(1/2)]$ is not quasitriangular and, furthermore, it is not almost cocommutative. In all other admissible cases the question about the existence of R is an open one.

The results we have announced in the present paper are more of a mathematical nature. The very fact, however, that $a^{\pm}$ are deformed para-Bose operators (in some other terminology, deformed supersingletons [39]) indicates already their relation to quantum physics. In fact the representation with $p = 1$ corresponds deformed to bosons [22–25]. The one-dimensional quantum oscillator based on such operators exhibits quite unusual properties when q is a root of 1. In particular it leads to discretization of the spectrum of the position and momentum operators, thus putting the phase space on a lattice [40]. It will be interesting to consider the same problem in the frame of the more general para-Bose oscillator, considering all its (unitarizable) root-of-1 representations.

Various kinds of oscillators based on deformed parabosons have so far been discussed in the literature (see [1] for references in this respect) without usually paying attention to the underlying coalgebra structure. The arbitrary deformations may face serious problems, however: if the underlying deformed para-Bose algebra is not a Hopf algebra (or at least an associative algebra with a comultiplication, which is an algebra morphism), it is impossible to define the tensor products of representations. The deformations of the parabosons considered here are free of this disadvantage, since our deformed algebra is identical with the Hopf algebra $U_q[osp(1/2)]$. Another positive feature of the Hopf algebra deformations is the existence of an R-matrix within every Fock space $W^L(p)$. The latter allows one to define an action of the braid group B_N within any Nth tensorial power $W^L(p)^{\otimes N}$, which commutes with $U_q[osp(1/2)]$. This is a step towards the decomposition of $W^L(p)^{\otimes N}$ into irreducible $U_q[osp(1/2)]$ modules.

It will be interesting to generalize the present approach to the case of several, say, n modes of pre-oscillator operators. To this end one has first to express the universal $U_q[osp(1/2n)]$ R-matrix in terms of deformed para-Bose operators and then consider root-of-1 representations of them. A good candidate for such a representation is that of the q-commuting deformed Bose operators, introduced recently in [20, 21], which permit only root-of-1 (unitary) representations.

Acknowledgments

All results of the present investigation were obtained during our three months visit at the International Centre for Theoretical Physic in Trieste. We are grateful to Professor Randjbar-Daemi for his invitation and for the kind hospitality of the High Energy Section of ICTP. We are thankful to Professor D Arnaudon for constructive remarks and suggestions. The work was supported by the Grant Φ-416 of the Bulgarian Foundation for Scientific Research.

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